

# One-Loop Renormalization of Lee-Wick Gauge Theory

Benjamín Grinstein<sup>1,\*</sup> and Donal O'Connell<sup>2,†</sup>

<sup>1</sup>*Department of Physics, University of California at San Diego, La Jolla, CA 92093*

<sup>2</sup>*Institute for Advanced Study, School of Natural Sciences, Einstein Drive, Princeton, NJ 08540*

(Dated: February 4, 2008)

We examine the renormalization of Lee-Wick gauge theory to one loop order. We show that only knowledge of the wavefunction renormalization is necessary to determine the running couplings, anomalous dimensions, and vector boson masses. In particular, the logarithmic running of the Lee-Wick vector boson mass is exactly related to the running of the coupling. In the case of an asymptotically free theory, the vector boson mass runs to infinity in the ultraviolet. Thus, the UV fixed point of the pure gauge theory is an ordinary quantum field theory. We find that the coupling runs more quickly in Lee-Wick gauge theory than in ordinary gauge theory, so the Lee-Wick standard model does not naturally unify at any scale. Finally, we present results on the beta function of more general theories containing dimension six operators which differ from previous results in the literature.

## I. INTRODUCTION

In recent months, an extension of the standard model of particle physics has been constructed [1] based on ideas of Lee and Wick [2]. Lee and Wick constructed a finite theory of quantum electrodynamics in order to remove divergences in certain mass corrections. The theory of Lee and Wick contains new degrees of freedom which are associated with wrong sign kinetic terms. Thus the theory is classically unstable. Lee and Wick proposed that the instability could be removed at the classical level by imposing boundary conditions on the theory, and at the quantum level by quantizing the theory such that the energy of any scattering (asymptotic) state is positive. This requires the introduction of a non-positive definite

---

\*bgrinstein@ucsd.edu

†donal@ias.edu

norm on the Hilbert space. Lee and Wick further described how the theory could nevertheless be unitary if the negative norm states are heavy and can decay to states of positive norm. These ideas have been discussed extensively in the literature [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. It has not been shown that an arbitrary Lee-Wick theory is unitary to all orders of perturbation theory, but there is no known example of a theory that cannot be unitarized in this way. In particular, scalar Lee-Wick theories have been extensively analyzed in [10] at the non-perturbative level with encouraging results.

With the modern understanding of renormalization the original motivation of Lee and Wick is no longer pressing, and in particular the massive resonances predicted by the Lee-Wick theory of electrodynamics have not been observed. Thus, interest in the Lee-Wick model of electrodynamics has dwindled. However, we are currently faced with quadratically divergent radiative corrections to the Higgs mass. The extension of the standard model developed in [1], known as the Lee-Wick standard model, includes new degrees of freedom that remove these quadratic divergences. The resulting theory is logarithmically divergent. The new degrees of freedom are associated with higher derivative, dimension six operators present in the microscopic Lagrangian of the theory. It was shown that an equivalent formulation of the theory contains only dimension four operators; in this form, the new degrees of freedom in the theory have wrong sign kinetic Lagrangians. The Lee-Wick prescription is then invoked to quantize the theory. Physically, the Lee-Wick standard model is unusual since the future boundary condition leads to acausality. However, the time scale of this acausality is far too small to have been ruled out by experiment.

The flavor structure of the Lee-Wick standard model has been explored in [12] with the attractive result that while new flavor changing neutral and charged currents are present, the flavor symmetry violation is naturally within experimental bounds. However, the Lee-Wick standard model was defined by choosing particular dimension six operators to add to the standard model Lagrangian. One could imagine a more general theory containing a greater number of dimension six operators. Some of these operators would lead to unacceptably large flavor changing currents. In [13] the question of the physical status of such operators was addressed, and it was shown that the choice of operators made in defining the Lee-Wick standard model is such that scattering amplitudes in the theory do not violate the well-known perturbative unitarity bounds. Thus, while dimension six operators typically imply either strong coupling in the ultraviolet or a violation of unitarity, the operators included

in the Lee-Wick standard model lead to a perturbative UV completion as suggested by precision electroweak constraints. Aspects of the LHC phenomenology of the Lee-Wick standard model has been discussed in [14, 15, 16], and more theoretical aspects of these models have been examined in [17] and in [18]. Supersymmetric models including similar higher dimension operators have been examined in [19].

In the present work, we turn to the question of the one-loop structure of non-abelian Lee-Wick gauge theory. A perturbative power counting argument presented in [1] establishes that the dimension six operators in the higher derivative formulation of the theory only receive finite renormalizations. In this work, we examine the renormalization in more detail. We work in background field gauge. There are some subtleties of gauge fixing in these theories which we discuss before turning our attention to the beta function and anomalous dimensions of matter. One interesting result is that the running of the massive vector boson mass,  $m$ , in the theory is exactly related to the running of the coupling,  $g$ , because the quantity  $mg$  is a renormalization group invariant. In an asymptotically free quantum field theory,  $g$  runs to zero in the ultraviolet so if  $mg \neq 0$ , then the mass  $m$  must run to infinity in the UV. Consequently the UV fixed point of the renormalization group flow is an ordinary free quantum field theory. We find that the Lee-Wick standard model does not appear to unify at any energy scale. The Lee-Wick particles in the theory in fact cause the running of the coupling to be quicker, so that any putative unification scale would be rather low. If the unification group were to be semisimple, this would lead to unacceptably large proton decay, but this problem can be alleviated [20]. We then turn to more general theories containing dimension six operators which are not of Lee-Wick type. While these theories do not satisfy the perturbative unitarity bounds, they have nevertheless been discussed in the literature as a toy model of higher derivative gravity [21, 22]. Since our results for the beta functions of these theories differ from previous expressions in the literature we feel it is worthwhile to present our results.

## II. PRELIMINARIES

The theory we study is given by<sup>1</sup>

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu}) + \frac{1}{m^2}\text{Tr}(D^\mu F_{\mu\nu})^2 + \bar{\psi}_L i \not{D} \psi_L + \frac{\sigma_1}{m^2} \bar{\psi}_L i \not{D} \not{D} \psi_L - \phi^* D^2 \phi - \frac{\delta_1}{m^2} \phi^* (D^2)^2 \phi. \quad (1)$$

Our notation is as follows.  $A_\mu^a$  is the gauge field with field strength  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \dots$ . In matrix notation  $F_{\mu\nu} = T^a F_{\mu\nu}^a$  and  $A_\mu = T^a A_\mu^a$  with  $T^a$  hermitian generators of the defining representation of the gauge group (traceless for factors of a semisimple group). The ‘Tr’ denotes a trace in the space of these matrices, the normalization is  $\text{Tr} T^a T^b = \frac{1}{2} \delta^{ab}$  and the structure constants are  $[T^a, T^b] = i f^{abc} T^c$ . The covariant derivative is  $D_\mu = \partial_\mu \mathbf{1} + ig A_\mu$ .  $\psi_L$  is a left handed spinor,  $\phi$  a complex scalar. Our metric convention is  $(+ - - -)$ .

This is not the most general gauge and Lorentz invariant Lagrangian with operators of dimension no larger than six, on two counts. First, we have omitted a scalar potential and Yukawa scalar-spinor interactions. There is no technical barrier to considering these, but our interest here is on the renormalization of the gauge sector. And secondly, we extended the renormalizable Lagrangian by three specific higher derivative terms. These are the only terms one may add such that the Lagrangian can be equivalently formulated as a renormalizable theory that includes additional negative metric Lee-Wick (LW) fields. We are particularly interested in this class of theories since it has been shown that for massive vector scattering they preserve perturbative unitarity, while this is not the case for theories with other type of dimension six derivative operators[13].

Consider the Lagrangian,

$$\begin{aligned} \mathcal{L}_{LW} = & -\frac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu}) + 2\text{Tr}(F_{\mu\nu}D^\mu \tilde{A}^\nu) - m^2\text{Tr}\tilde{A}^\mu \tilde{A}_\mu \\ & + \bar{\psi}_L i \not{D} \psi_L + \bar{\tilde{\psi}}_L i \not{D} \tilde{\psi}_L + \bar{\psi}_L i \not{D} \tilde{\psi}_L - \bar{\tilde{\psi}}_R i \not{D} \tilde{\psi}_R + \frac{m^2}{\sigma_1} \left( \bar{\tilde{\psi}}_L \tilde{\psi}_R + \bar{\tilde{\psi}}_R \tilde{\psi}_L \right) \\ & - \phi^* D^2 \phi - \tilde{\phi}^* D^2 \phi - \phi^* D^2 \tilde{\phi} + \frac{m^2}{\delta_1} \tilde{\phi}^* \tilde{\phi}. \quad (2) \end{aligned}$$

Upon solving the equations of motion of the fields  $\tilde{A}_\mu$ ,  $\tilde{\psi}$  and  $\tilde{\phi}$  and inserting the solutions in (2) one recovers the higher derivative Lagrangian of (1). While the Lagrangian (2) has twice

---

<sup>1</sup> We have only written one chiral fermion, which would lead to an anomaly in the gauge symmetry. This is for simplicity; the potential anomaly will play no role in our work. In our computations below we will initially discuss the contribution of one fermion before generalizing to arbitrary matter content.

as many fields as the higher derivative version (1) it is renormalizable by power counting, so it is more convenient to use in some cases. The mixing terms present in (2) can be diagonalized by an appropriate redefinition of the fields, as discussed in [1].

For our calculations below we use the background field gauge (BFG) method. Let us briefly review it. This is not only for completeness: as we shall shortly show, one has to be careful about introducing higher derivatives in the gauge fixing term. Denote the quantum fields by  $A_\mu$  and the background fields by  $B_\mu$ . The effective action is determined by the vacuum graphs for the theory with action integral  $S(A + B)$ , where  $S = \int d^4x \mathcal{L}$ , and  $\mathcal{L}$  as given above. The gauge fixing condition is

$$\mathcal{F}(A, B) = 0 \quad (3)$$

for some function that is invariant under gauge transformations of the  $B$  field with the  $A$  field transforming as a matter field:

$$B_\mu \rightarrow U \left( \frac{1}{ig} \partial_\mu + B_\mu \right) U^\dagger \quad (4)$$

$$A_\mu \rightarrow U A_\mu U^\dagger. \quad (5)$$

The simplest covariant choice is

$$\mathcal{F}(A, B) = D(B)_\mu A^\mu = \partial_\mu A^\mu + ig[B_\mu, A^\mu]. \quad (6)$$

By shifting  $A \rightarrow A - B$  the BFG method can be understood in terms of the formulation in the absence of a background field but with a  $B$ -dependent gauge fixing condition,

$$\mathcal{F}(A - B, B) = 0. \quad (7)$$

The Faddeev-Popov determinant  $\Delta_{FP}$  can be computed through the ghost Lagrangian

$$\mathcal{L}_{FP} = \bar{b} [D(B)^\mu D(A + B)_\mu] c, \quad (8)$$

where  $b$  and  $c$  are the anti-commuting scalars in the adjoint representation.

The gauge fixing condition can be brought into the action in the functional integral by the usual trick: writing the condition as  $\delta(\mathcal{F} - \alpha)$ , one then uses the averaged partition function

$$Z = \int [d\alpha] \exp \left( \frac{-i}{2\xi} \int d^4x \alpha^2 \right) Z_\alpha, \quad (9)$$

where

$$Z_\alpha = \int [dA] e^{iS} \Delta_{FP} \delta(\mathcal{F} - \alpha). \quad (10)$$

It is sometimes useful to have more derivatives in the gauge fixing term in the action (for example, for power counting arguments). This can be done by putting derivatives in the exponent in the exponentiation trick (9). However we must be careful to preserve the invariance in (4)–(5). So an alternative form of the partition function we may use is

$$Z = \sqrt{\det \left( 1 + \frac{1}{M^2} D(B)^2 \right)} \int [d\alpha] \exp \left( \frac{-i}{2\xi} \int d^4x \alpha \left[ 1 + \frac{1}{M^2} D(B)^2 \right] \alpha \right) Z_\alpha. \quad (11)$$

Notice the factor of the square root of the determinant, which compensates for the extraneous  $B$  dependence introduced by the  $\alpha$  integration. Below we compute the beta functions of this theory with both types of gauge fixing and find agreement.

The determinant in (11) can be computed using ghost fields,

$$\det(1 + \frac{1}{M^2} D(B)^2) = \int [db][dc] e^{i \int d^4x \bar{b} (-D(B)^2 - M^2) c}. \quad (12)$$

The Lagrangian for these ghosts is similar to that of the Faddeev-Popov ghosts, except that this one has a mass and lacks a coupling to the quantum field. This observation is useful because as far as the computation of the infinite part of the  $B$  self-energy is concerned there is no difference between this and the Faddeev-Popov case. So these ghosts contribute to the infinite part of the self-energy one half of the FP ghosts, the factor of one half arising from taking square root of the determinant.

One last comment is in order before we embark on our computation. To properly construct the S-matrix for a Lee-Wick theory one needs to adopt specific prescriptions for the choice of contours in Feynman diagrams. As a result, there are well known difficulties in writing a functional integral version of the quantization of the theory and no consensus on whether a functional integral version exists; see [9, 10, 11]. The above discussion on the BFG method uses extensively the functional integral formalism. This can be easily justified. To the extent that we are only interested in renormalization, that is, in the ultraviolet divergences of the theory, the detailed choice of integration contours is irrelevant. The difference between any two integration contours in the complex energy plane in a Lee-Wick amplitude gives always a residue at a pole, and is therefore finite (even after integrating over spatial momentum).

### III. RENORMALIZATION

The renormalized version of the Lagrangian (1) is

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}Z\text{Tr}(F^{\mu\nu}F_{\mu\nu}) + Z_\psi\bar{\psi}_L i\not{D}\psi_L - Z_\phi\phi^*D^2\phi + \frac{1}{m^2}ZZ_{m^2}[\text{Tr}(D^\mu F_{\mu\nu})^2] \\ & + \frac{1}{m^2}Z_\psi Z_{m^2}(Z_\sigma\sigma_1)\bar{\psi}_L[i\not{D}\not{D}\not{D}]\psi_L - \frac{1}{m^2}Z_\phi Z_{m^2}(Z_\delta\delta_1)\phi^*[(D^2)^2]\phi. \end{aligned} \quad (13)$$

The first line contains the kinetic terms (dimension four operators) and it is in terms of these that the wave function renormalization factors  $Z$ ,  $Z_\psi$  and  $Z_\phi$  are defined. The next three lines contain the dimension six operators for gauge fields, spinors and scalars, respectively. The coupling constant renormalization is not shown explicitly, but it should be understood that the Lagrangian depends on  $g$  through the combination  $Z_g g$  only.

Some comments are in order. There are no counterterms of the form of any of the dimension six operators, a result that was established in [1] by the power counting analysis and verified through an explicit one loop computation. This implies for example that  $ZZ_{m^2}$  is finite, so we can adopt the renormalization condition

$$ZZ_{m^2} = 1. \quad (14)$$

Similarly, we have

$$Z_\psi Z_{m^2} Z_\sigma = Z_\phi Z_{m^2} Z_\delta = 1. \quad (15)$$

We have chosen to work in background field gauge. One of the great simplifications of BFG is that [23]

$$Z_g Z^{\frac{1}{2}} = 1. \quad (16)$$

With this and Eqs. (14)–(15) we deduce that the full set of renormalization constants is given in terms of the three wavefunction renormalization constants.

To write the Renormalization Group Equations (RGE) we Taylor expand the renormalization constants with respect to  $\epsilon \equiv 4 - D$ . Define residues  $a$  through

$$Z = 1 + \frac{a}{\epsilon} + \dots \quad (17)$$

$$Z_g = 1 + \frac{a_g}{\epsilon} + \dots \quad (18)$$

$$\vdots \quad (19)$$

$$Z_\delta = 1 + \frac{a_\delta}{\epsilon} + \dots \quad (20)$$

Then, as usual,

$$\beta(g, \epsilon) = -\frac{1}{2} \epsilon g + \beta(g), \quad \beta(g) = \frac{1}{2} g^2 \frac{\partial a_g}{\partial g}. \quad (21)$$

Using, from (16),  $a_g = -\frac{1}{2}a$  and putting together the contributions to the YM self energy in the previous section we have,

$$\beta(g) = -\frac{1}{4} g^2 \frac{\partial a}{\partial g}. \quad (22)$$

The anomalous dimensions for the matter fields are

$$\gamma_f(g) = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \log Z_f = -\frac{1}{4} g \frac{\partial a_f}{\partial g}, \quad f = \psi, \phi. \quad (23)$$

The renormalization group equation for the matter couplings is easily obtained. We present this for a single spinor or scalar, to avoid unnecessary complications from the matrix structure:

$$\mu \frac{\partial \sigma_1}{\partial \mu} = -\sigma_1 \gamma_{\sigma_1}(g) = 2\sigma_1 \left( \gamma_\psi(g) - \frac{\beta(g)}{g} \right) \quad (24)$$

$$\mu \frac{\partial \delta_1}{\partial \mu} = -\delta_1 \gamma_{\delta_1}(g) = 2\delta_1 \left( \gamma_\phi(g) - \frac{\beta(g)}{g} \right), \quad (25)$$

or more simply

$$\mu \frac{\partial (g^2 \sigma_1)}{\partial \mu} = 2(g^2 \sigma_1) \gamma_\psi(g) \quad \text{and} \quad \mu \frac{\partial (g^2 \delta_1)}{\partial \mu} = 2(g^2 \delta_1) \gamma_\phi(g). \quad (26)$$

We turn now to the explicit computation of the self-energy diagrams.

### A. YM self-energy

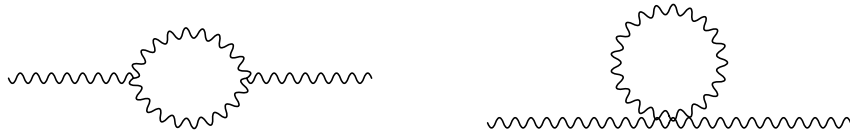


FIG. 1: Contribution to the self-energy of YM fields from internal YM fields

As mentioned above, we have performed the computation several different ways: we can use a higher derivative version of the theory with a standard covariant gauge fixing term, or we can use a higher derivative version of the covariant gauge fixing term with a Jacobian correction, or we can use the formulation of the theory without higher derivative terms but instead including negative norm LW fields. In each case the computation is very different.



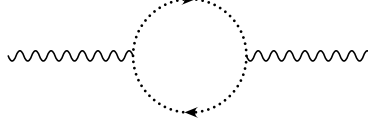


FIG. 2: Contribution to the self-energy of YM fields from internal ghosts

There is no one to one correspondence between the contributions to the renormalization constants of individual Feynman diagrams, yet the resulting beta functions are the same.

We first list our results for the  $\epsilon$ -poles of the graphs computed in the higher derivative theory with a standard covariant BFG-term. The graphs in Figs.1 give

$$\frac{ig^2}{16\pi^2}\delta^{ab}\left(\frac{2}{\epsilon}\right)\frac{41}{6}C_2(g_{\mu\nu}k^2 - k_\mu k_\nu), \quad (27)$$

where  $C_2$  is defined by  $\sum_{x,y} f^{axy} f^{bxy} = C_2 \delta^{ab}$ . The ghost graph in Fig. 2 yields a contribution

$$\frac{ig^2}{16\pi^2}\delta^{ab}\left(\frac{2}{\epsilon}\right)C_2\left(\frac{1}{3}\right)(g_{\mu\nu}k^2 - k_\mu k_\nu). \quad (28)$$

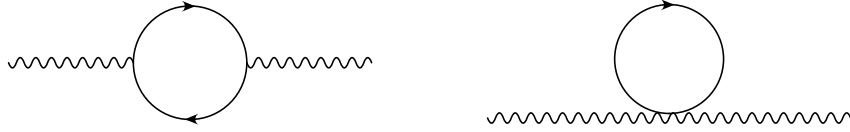


FIG. 3: Contribution to the self-energy of YM fields from internal spinor fields

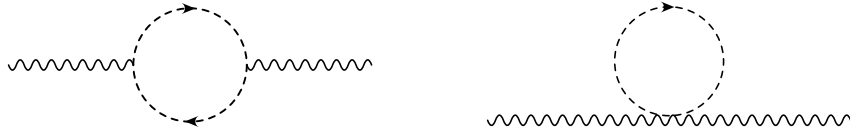


FIG. 4: Contribution to the self-energy of YM fields from internal complex scalar fields

Next come the matter fields. The spin-1/2 contribution (in the fundamental representation of the gauge group) from Figs. 3 is

$$-\frac{ig^2}{16\pi^2}\delta^{ab}\left(\frac{2}{\epsilon}\right)(g_{\mu\nu}k^2 - k_\mu k_\nu). \quad (29)$$

Finally, the contribution from a complex scalar field (in the fundamental representation) in Figs. 4 is given by

$$-\frac{ig^2}{16\pi^2}\delta^{ab}\left(\frac{2}{\epsilon}\right)\frac{1}{3}(g_{\mu\nu}k^2 - k_\mu k_\nu). \quad (30)$$

Now we turn to the case where we use a higher derivative version of the covariant BFG-term. The only difference from the above is in the graphs in Figs.1 which now give

$$\frac{ig^2}{16\pi^2}\delta^{ab}\left(\frac{2}{\epsilon}\right)\frac{40}{6}C_2(g_{\mu\nu}k^2 - k_\mu k_\nu). \quad (31)$$

However, now must also include a factor to compensate for the background field dependence of the modified measure, see Eq. (11). The determinant can be computed using a ghost as explained in Sec. II, and the result is therefore 1/2 of the usual ghost contribution of (28), namely

$$\frac{ig^2}{16\pi^2}\delta^{ab}\left(\frac{2}{\epsilon}\right)\frac{1}{6}C_2(g_{\mu\nu}k^2 - k_\mu k_\nu). \quad (32)$$

The sum of these two contributions precisely equals the result in (27).

Finally, we have computed the beta function in the Lee-Wick formulation of the theory, as discussed in [1]. In this formulation, the physical degrees of freedom are the gauge fields  $A_\mu^a$ , massive LW vector fields  $\tilde{A}_\mu^a$ , a chiral spin 1/2 field, a Dirac Lee-Wick fermion, a scalar field and a Lee-Wick scalar field. We compute the beta function by computing the wavefunction renormalization of the normal gauge fields in background field gauge.

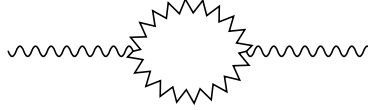


FIG. 5: Contribution to the self-energy of YM fields from internal LW-vector fields

It is easy to deduce the contributions of the matter fields to the beta function, because the LW fields couple to the gauge fields just as normal fields do<sup>2</sup>. Thus, the total contribution of the spin-1/2 fields to the beta function is three times the usual contribution of a fundamental chiral spin-1/2 fermion, while the scalar fields contribute twice the usual scalar field value, in agreement with Eq. (29) and Eq. (30), respectively. It remains to compute the effects of the LW vector fields. The relevant graph is shown in Fig. 5. The graph evaluates to<sup>3</sup>

$$\frac{ig^2}{16\pi^2}\delta^{ab}\left(\frac{2}{\epsilon}\right)\frac{7}{2}C_2(g_{\mu\nu}k^2 - k_\mu k_\nu). \quad (33)$$

<sup>2</sup> Signs associated with Lee-Wick propagators appear squared in all the relevant diagrams.

<sup>3</sup> In this formulation of the theory, there are additional divergences proportional to  $p^4$  and  $p^6$  which we do not show. These higher divergences are gauge artifacts. Since the beta function is gauge independent to this order, we can be confident of our results. It is possible to fix the gauge in the Lee-Wick formulation of the theory so that these spurious divergences do not appear, at the expense of a more involved formalism.

Of course, the gauge fields and ghost lead to a term

$$\frac{ig^2}{16\pi^2}\delta^{ab}\left(\frac{2}{\epsilon}\right)\frac{11}{3}C_2(g_{\mu\nu}k^2 - k_\mu k_\nu). \quad (34)$$

in the beta function; adding this value to (33) again equals the result of (27).

### B. Matter self-energies



FIG. 6: Feynman diagrams contributing to the self-energy of spin-1/2 fields

Now we turn to the self-energies of matter fields. The spin-1/2 self energy is from Fig. 6. We find the divergent terms cancel among the two graphs. This result is the same for the higher derivative theory, with either type of gauge fixing, as for the LW fields version of the theory. The self-energy of the complex scalar from the sum of the diagrams in Fig. 7 gives



FIG. 7: Feynman diagrams contributing to the self-energy of spin-0 fields

a wavefunction renormalization

$$\frac{g^2}{16\pi^2}\left(\frac{2}{\epsilon}\right)6C_1\delta_1ik^2 \quad (35)$$

where  $C_1$  is defined by  $T^a T^a = C_1 \mathbf{1}$ . There is of course also mass renormalization but recall we have postponed the study of the renormalization of terms in the scalar potential.

## IV. BETA FUNCTION AND ANOMALOUS DIMENSIONS

Our final results are in the form of explicit expressions for the beta functions and anomalous dimensions. These are obtained combining the results above. First, the running of the

gauge coupling is determined by

$$\beta(g) = -\frac{g^3}{16\pi^2} \left[ \frac{43}{6}C_2 - n_f - \frac{1}{3}n_s \right]. \quad (36)$$

We have introduced  $n_f$  and  $n_s$  for the number of spinor and scalar fields. More generally, if the spin 1/2 and 0 fields are in arbitrary representations of the gauge group we have

$$\beta(g) = -\frac{g^3}{16\pi^2} \left[ \frac{43}{6}C_2 - 2 \sum_f n_f T(f) - \frac{2}{3} \sum_s n_s T(s) \right], \quad (37)$$

where in the representation  $x$  we have  $\text{Tr}(T^a T^b) = T(x)\delta^{ab}$ .

Similarly, the anomalous dimensions for the spinor and scalar are

$$\gamma_\psi(g) = 0, \quad (38)$$

and

$$\gamma_\phi(g) = -\frac{g^2}{16\pi^2} 3C_1 \delta_1. \quad (39)$$

Combining (14) and (16) we see that  $Z_{m^2} = Z_g^2$ , so the solution to the RGE for  $m^2$  is immediate,

$$m^2(\mu) = \left( \frac{g^2(\mu_0)}{g^2(\mu)} \right) m^2(\mu_0). \quad (40)$$

Since  $\gamma_\psi(g) = 0$ , we see from Eq. (26) that  $g^2\sigma_1$ , and so  $m^2/\sigma_1$  does not run. Therefore, the mass of the Lee-Wick fermion is an invariant of the RG flow. On the other hand, the quantity  $g^2\delta_1$  does flow so that the LW scalar mass grows logarithmically in the ultraviolet.

The result (37) is roughly what one would guess naively. The higher derivative terms that we have introduced in the Lagrangian are precisely the ones that can be described as additional LW fields. Hence one roughly expects to double the contribution of each field to the  $\beta$ -function. Subtleties occur in the spinor matter and pure gauge terms. In the spinor terms, the contribution is tripled because the Lee-Wick partner of a chiral fermion is non-chiral. In the pure gauge term, the contribution from ghosts is not quite doubled as explained above.

Hence, much like for the standard model of electroweak interactions, the LW extension of the standard model does not display good unification of coupling constants. The standard model does however unify well if properly chosen additional fields are introduced. A simple example was given by Willenbrock in Ref. [20], where he shows that the standard model with six Higgs doublets unifies. Similarly, we find that the Lee-Wick extension of the standard

model has good coupling constant unification if it is extended to include six or seven Higgs doublets.

However, Willenbrock points out that in the six-Higgs doublet model the unification scale is very low so if the unification group is simple, then the proton decays excessively fast. He proposes an interesting solution to this problem using trinification, that is, a unified group  $SU(3)^3/Z_3$ . The unification scale in our six Higgs doublet model is even lower than in Willenbrock's case, about a million times the LW scale  $m$ . Presumably one can formulate a LW extension of trinification, but we have not pursued this.

## V. ADDITIONAL DIMENSION SIX OPERATORS

In this section we consider a more general theory which contains additional dimension six operators in the Lagrangian density. This theory does not satisfy the constraints of perturbative unitarity, so that scattering amplitudes cannot be computed by perturbative methods. The beta function and anomalous dimensions, on the other hand, may still be computed in perturbation theory since no large energies occur in these functions. Theories of these types have been considered in the literature previously as toy models for higher derivative gravity [21, 22]. The Lagrangian of the theory is given by

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_\psi + \mathcal{L}_\phi, \quad (41)$$

where

$$\mathcal{L}_A = -\frac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu}) + \frac{1}{m^2}\text{Tr}(D^\mu F_{\mu\nu})^2 - \frac{i\gamma g}{m^2}\text{Tr}(F^{\mu\nu}[F_{\mu\lambda}, F_\nu{}^\lambda]) \quad (42)$$

specifies the dynamics of the gauge sector. The spinor matter Lagrangian is

$$\mathcal{L}_\psi = \bar{\psi}_L i \not{D} \psi_L + \frac{i}{m^2} \bar{\psi}_L [\sigma_1 \not{D} \not{D} \not{D} + \sigma_2 \not{D} D^2 + ig\sigma_3 F^{\mu\nu} \gamma_\nu D_\mu + ig\sigma_4 (D_\mu F^{\mu\nu}) \gamma_\nu] \psi_L, \quad (43)$$

where, in the last term of the Lagrangian, the covariant derivative acts only on the field strength tensor, and  $\sigma_{1-4}$  are dimensionless constants. For a complex scalar, we consider for the Lagrangian density

$$\mathcal{L}_\phi = -\phi^* D^2 \phi - \frac{1}{m^2} \phi^* [\delta_1 (D^2)^2 + ig\delta_2 (D_\mu F^{\mu\nu}) D_\nu + g^2 \delta_3 F^{\mu\nu} F_{\mu\nu}] \phi, \quad (44)$$

where, as above, the parenthesis in the second term indicates that the derivative to the left of  $F_{\mu\nu}$  acts only on  $F_{\mu\nu}$ .

We find that the beta function and anomalous dimensions are given by

$$\beta(g) = -\frac{g^3}{16\pi^2} \left[ \left( \frac{43}{6} - 18\gamma + \frac{9}{2}\gamma^2 \right) C_2 - n_\psi \left( \frac{\sigma_1^2 - \sigma_2\sigma_3 + \frac{1}{2}\sigma_3^2}{(\sigma_1 + \sigma_2)^2} \right) - n_\phi \left( \frac{\delta_1 + 6\delta_3}{3\delta_1} \right) \right], \quad (45)$$

$$\gamma_\psi(g) = -\frac{g^2}{16\pi^2} \frac{3}{4} C_1 \left( \frac{2\sigma_1(2\sigma_2 + \sigma_3 - 2\sigma_4) + \sigma_2(2\sigma_2 + 2\sigma_3 - \sigma_4) - \sigma_3^2 - \sigma_4^2 + \sigma_3\sigma_4}{\sigma_1 + \sigma_2} \right), \quad (46)$$

$$\gamma_\phi(g) = -\frac{g^2}{16\pi^2} \frac{3}{8} C_1 \left( \frac{8\delta_1^2 - \delta_2^2 - 4\delta_1\delta_2}{\delta_1} \right). \quad (47)$$

We note that our expression for the beta function differs from that found in Appendix C of [22]. We can write the beta functions for the couplings of the dimension six operators in terms of these anomalous dimensions. The first states that  $\gamma$  is a constant,

$$\mu \frac{\partial \gamma}{\partial \mu} = 0. \quad (48)$$

The equations for  $\sigma_i$  and  $\delta_i$  are the same as we found for  $\sigma_1$  and  $\delta_1$  in the previous section,

$$\mu \frac{\partial(g^2\sigma_i)}{\partial \mu} = 2(g^2\sigma_i)\gamma_\psi(g) \quad \text{and} \quad \mu \frac{\partial(g^2\delta_i)}{\partial \mu} = 2(g^2\delta_i)\gamma_\phi(g). \quad (49)$$

In particular, we see that the ratios  $\sigma_i/\sigma_j$  and  $\delta_i/\delta_j$  do not run.

Clearly the renormalization group of this theory is much richer than the one considered in the previous section. In particular, for the theory based on the standard model (that is, the theory which has the same field content as the standard model), there is now an additional free parameter that enters the scale of unification, namely the cubic field strength coupling of the unified theory,  $\gamma$ . One can in fact have successful  $SU(5)$  unification in this theory, with a unification scale in excess of  $10^{16}m$  for  $0.33 \lesssim \gamma \lesssim 0.35$  or  $3.65 \lesssim \gamma \lesssim 3.67$ . While this may seem phenomenologically adequate, we remind the reader that this theory is not perturbatively unitary.

Another interesting property of the result in (45) is that the coupling constants,  $\gamma$ ,  $\sigma_i$  and  $\delta_i$  can be chosen to make the  $\beta$ -function vanish. Note that the quantity in square brackets in (45) is renormalization group invariant so one may consistently set it to any fixed value.

## VI. CONCLUDING REMARKS

Typically, Lagrangian densities in particle physics which contain operators of dimension higher than four lead to theories which are less predictive. This is a result of the divergences introduced by these operators in perturbation theory. New counterterms must be introduced

to absorb these divergences, typically leading to theories containing an infinite number of couplings constants, which are a priori unknown.

The situation is different in Lee-Wick theories. In the higher derivative formulation of the theories, dimension six operators are present in the microscopic Lagrangian. There is one new constant associated with each higher derivative operator, which physically corresponds to the mass of the corresponding Lee-Wick degree of freedom.

In this work, we have described the renormalization of Lee-Wick theories to one-loop order. No new counterterms are required to absorb the divergences of the theory. In fact, we have shown that the wavefunction renormalizations of the various fields present in Lee-Wick gauge theory contain all the information about the renormalization group running of the theory. For the Lee-Wick gauge bosons, we have shown that the quantity  $m^2 g^2$  is an invariant of the renormalization flow. Thus, the new constant introduced in the definition of a Lee-Wick gauge theory truly is just one number, and not a new function of energy scale. In addition, we learn that if the theory is asymptotically free, then the LW vector boson mass flows to infinity in the UV. This counter intuitive behaviour is interesting because it indicates that the ultraviolet fixed point of the RG flow of an asymptotically free pure Lee-Wick gauge theory is a normal quantum field theory: the scale suppressing the dimension six operator in the higher dimension formulation of the theory has become infinite so that this term no longer contributes to the dynamics. The remaining degrees of freedom are the usual gauge bosons.

We have obtained expressions for the beta function and anomalous dimensions of scalar and spinor matter. The coupling runs more quickly in Lee-Wick theory compared to the usual non-Abelian gauge theory. We find that the Lee-Wick standard model does not unify naturally, and that, on account of the more rapid running of the coupling, the unification scale of the theory augmented with extra field content is typically rather low. In addition, we find that the anomalous dimension of spinor matter vanishes.

Finally, we have discussed some more general theories containing dimension six operators which are not of Lee-Wick type. Since amplitudes in these theories grow too quickly with energy to satisfy perturbative unitarity bounds, the theories either become non-perturbative at some scale, or else they violate unitarity. However, no large factors of energy appear in the expressions for the beta function or for the anomalous dimensions, so they may still be computed in perturbation theory. (Of course, they no longer give us insight into the high

energy behaviour of physical scattering amplitudes.) These theories have been discussed elsewhere in the literature, and since our results differ from previous expressions we have reported our results above. Our results indicate that if it is possible to make sense of these theories, then, for suitable choices of the couplings, these theories may enjoy the property that their beta function vanishes.

### Acknowledgments

We thank Mark B. Wise for useful discussions and collaboration at the beginning of this work, and we are grateful to Arkady A. Tseytlin for email correspondence. DOC thanks Poul Henrik Damgaard for several helpful conversations. The work of BG and DOC was supported in part by the US Department of Energy under contracts DE-FG03-97ER40546 and DE-FG02-90ER40542, respectively.

- 
- [1] B. Grinstein, D. O’Connell and M. B. Wise, arXiv:0704.1845 [hep-ph].
  - [2] T. D. Lee and G. C. Wick, Nucl. Phys. B **9**, 209 (1969); T. D. Lee and G. C. Wick, Phys. Rev. D **2**, 1033 (1970).
  - [3] S. Coleman, In “*Erice 1969, Ettore Majorana School On Subnuclear Phenomena*”, New York, 282 (1970).
  - [4] R. E. Cutkosky, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, Nucl. Phys. B **12**, 281 (1969).
  - [5] N. Nakanishi, Phys. Rev. D **3**, 811 (1971).
  - [6] T. D. Lee and G. C. Wick, Phys. Rev. D **3**, 1046 (1971).
  - [7] N. Nakanishi, Phys. Rev. D **3**, 1343 (1971).
  - [8] I. Antoniadis and E. T. Tomboulis, Phys. Rev. D **33**, 2756 (1986).
  - [9] D. G. Boulware and D. J. Gross, Nucl. Phys. B **233**, 1 (1984).
  - [10] K. Jansen, J. Kuti and C. Liu, Phys. Lett. B **309**, 119 (1993) [arXiv:hep-lat/9305003];  
K. Jansen, J. Kuti and C. Liu, Phys. Lett. B **309**, 127 (1993) [arXiv:hep-lat/9305004],  
Z. Fodor, K. Holland, J. Kuti, D. Nogradi and C. Schroeder, arXiv:0710.3151 [hep-lat].
  - [11] A. van Tonder, Int. J. Mod. Phys. A **22**, 2563 (2007) [arXiv:hep-th/0610185].



- [12] T. R. Dulaney and M. B. Wise, arXiv:0708.0567 [hep-ph].
- [13] B. Grinstein, D. O'Connell and M. B. Wise, arXiv:0710.5528 [hep-ph].
- [14] T. G. Rizzo, JHEP **0706**, 070 (2007) [arXiv:0704.3458 [hep-ph]].
- [15] F. Krauss, T. E. J. Underwood and R. Zwicky, arXiv:0709.4054 [hep-ph].
- [16] T. G. Rizzo, arXiv:0712.1791 [hep-ph].
- [17] F. Wu and M. Zhong, arXiv:0705.3287 [hep-ph].
- [18] J. R. Espinosa, B. Grinstein, D. O'Connell and M. B. Wise, arXiv:0705.1188 [hep-ph].
- [19] I. Antoniadis, E. Dudas and D. M. Ghilencea, arXiv:0708.0383 [hep-th].
- [20] S. Willenbrock, Phys. Lett. B **561**, 130 (2003) [arXiv:hep-ph/0302168].
- [21] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B **104**, 377 (1981).
- [22] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. B **201**, 469 (1982).
- [23] L. F. Abbott, Nucl. Phys. B **185**, 189 (1981).